NOTATION

a_k = kth weighting factors in filter equation
c_N = Nth filter output
p = exponential filter weighting factor
q = process noise covariance
r = measurement noise covariance
t_i = ith dynamic time period
z_N = nth observation

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Mechanical Equilibrium for Eccentric Rotating Disks

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Dynamic mechanical measurements can be very helpful in characterizing the molecular structure of polymer melts and solutions. Generally dynamic properties have been determined from a sinusoidally varying deformation (Ferry, 1970). To evaluate the dynamic material properties it is necessary to determine the in phase and quadrature components of the stress in reference to the sinusoidal deformation. Measuring these components can be difficult.

Gent (1960) has suggested an alternate method for dynamic shear measurements, the flow between two parallel disks rotating about noncoincident axes. This eccentric rotating disk (ERD) geometry, shown in Figure 1, gives forces in the x and y directions which are steady in time and yield the dynamic quantities η' and G' directly. Recently we have shown that these correlate very well with results of sinusoidal tests (Macosko and Davis, 1972).

Previous theoretical analyses have assumed that both the upper and lower disk rotate with the same angular velocity (Blyler and Kurtz, 1967; Bird and Harris, 1968; and others reviewed by Macosko and Davis, 1972). However, in all experimental devices reported, one disk is driven and the other follows through viscous drag. This note provides an analysis of the actual experimental situation and tests this analysis experimentally.

If the lower disk rotates at the same angular velocity as the upper, driven disk ($\Omega_1 = \Omega_2$ in Figure 1), then the stresses produced by the flow will cause forces F_{x1} and F_{x2} to act on the axes of lower and upper disk, respectively. Thus a net torque, aF_{x2} will act about the axis of the lower disk. If the lower disk is free to rotate it will accelerate unless this torque is balanced. To balance this torque we suggest that the lower disk rotates at a slightly smaller angular velocity, causing a small torsional flow to occur, generating an opposite torque. Additional torsional flow can occur due to any drag in the lower bearing.

NEWTONIAN FLUID

We can solve the Navier Stokes equations for the flow between ERD with the boundary conditions of solid body rotation at each disk surface:

$$\begin{aligned} &\text{at} \quad z = 0 \quad v_x = -\,\Omega_1 y \qquad v_y = \Omega_1 x \quad v_z = 0 \\ &\text{at} \quad z = h \quad v_x = -\,\Omega_2 y + \Omega_2 a \quad v_y = \Omega_2 x \quad v_z = 0 \end{aligned} \tag{1}$$

The solution is simplest in cartesian coordinates and generally follows that of Abbott and Walters (1970) and Goldstein (1971). For a fluid with zero density we find the following kinematics:

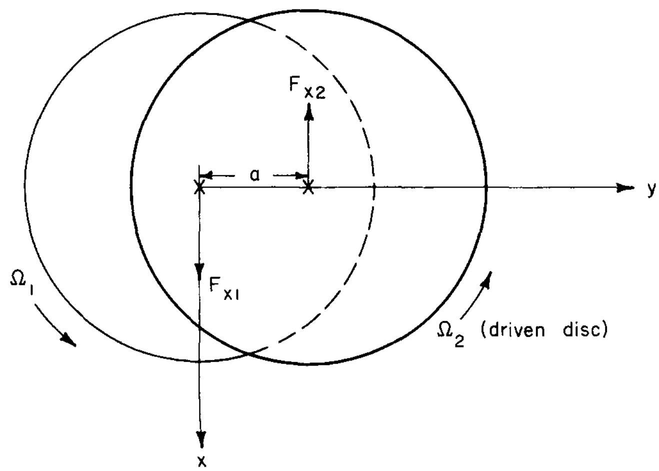

Fig. 1. Schematic diagram of ERD (top view), origin of cartesian coordinates is on lower disc.

$$v_x = -y[\Omega_1 + \underline{\Delta\Omega(z/h)}] + z\Omega_2(a/h)$$
$$v_y = x[\Omega_1 + \underline{\Delta\Omega(z/h)}] \quad (2)$$
$$v_z = 0$$

with the isotropic pressure equal to a constant.

This velocity distribution is just the ERD kinematics for $\Omega_1 = \Omega_2$ proposed by Blyler and Kurtz superimposed on torsional flow between concentric parallel disks. The underlined terms above are those due to torsional flow.

Shear stresses for the Newtonian fluid in this flow are

$$\tau_{xz} = \eta_0[\Omega_2(a/h) - \Delta\Omega(y/h)]$$
$$\tau_{yz} = \eta_0\,\Delta\Omega(x/h) \quad (3)$$

The torsional components of these stresses generate a torque within the fluid which acts on the lower disk M_i:

$$M_i = \int_0^{2\pi}\int_0^R r(\tau_{\phi z})_{z=0}\,(rdr)\,d\phi$$
$$= \int_0^{2\pi}\int_0^R r(-\tau_{xy}\sin\phi + \tau_{yz}\cos\phi)_{z=0}\,rdr\,d\phi \quad (4)$$
$$= \pi\,\Delta\Omega(R^4/2h)\eta_0$$

This internal torque is balanced by the two torques acting external to the fluid, one due to the eccentricity and the other to lower bearing friction. Thus

$$M_i = M_e = a\,F_{x2} + K\,\Omega_1 \quad (5)$$

where K is the drag constant of the air bearing and is determined experimentally. Solving for F_{x2}

$$F_{x2} = \int_0^{2\pi}\int_0^{R+a\sin\phi} (\tau_{xz})_{z=h}\,rdr\,d\phi$$
$$= \int_0^{2\pi}\int_0^{R+a\sin\phi} [\Omega_2(a/h) - \Delta\Omega(y/h)]rdr\,d\phi \quad (6)$$
$$= \pi\,R^2\,\Omega_1(a^2/h)\eta_0$$

where $y' = a + y = a + r\sin\phi$.

Substituting into (5) and rearranging gives

$$\frac{\Delta\Omega}{\Omega_1} = 2(a/R)^2 + \frac{2Kh}{\pi R^4\eta_0} \quad (7)$$

It is desirable to express the lag $\Delta\Omega$ in terms of the known velocity Ω_2 of the driven disk:

$$\frac{\Delta\Omega}{\Omega_2} = \frac{2(a/R)^2 + 2Kh/(\pi R^4\eta_0)}{1 + 2(a/R)^2 + 2Kh/(\pi R^4\eta_0)} \quad (8)$$

Under normal experimental conditions the numerator is much less than one so the most useful experimental relation for a Newtonian fluid is

$$\frac{\Delta\Omega}{\Omega_2} = 2(a/R)^2 + 2Kh/(\pi R^4\eta_0) \quad (9)$$

Thus the lag only depends on the ratio a/R and friction in the lower bearing.

VISCOELASTIC FLUID

A simple integral model (Lodge, 1964) was used to examine the effects of viscoelasticity:

$$\underline{\underline{\tau}} = \int_0^\infty m(\theta, 0)\underline{\underline{S}}\,d\theta \quad (10)$$

The components of the strain tensor in cartesian coordinates are

$$S_{ij} = \frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_k} - \delta_{ij}$$

where X = position coordinate of a fluid element at time t' and x = position at time t. This model is similar to the Bird Carreau model used by Bird and Harris, Gordon and Schowalter (1970) and Goldstein in their studies of the ERD geometry. The model has been successful in predicting small strain oscillatory behavior.

The displacement relationships for the kinematics [Equation (2)] are

$$x = X\cos\Omega_1(1 + \beta Z/h)\theta - \left[Y - \frac{(1+\beta)\gamma Z}{(1+\beta Z/h)}\right]\sin\Omega_1(1+\beta Z/h)\theta$$
$$y = X\sin\Omega_1(1+\beta Z/h)\theta + \left[Y - \frac{(1+\beta)\gamma Z}{(1+\beta Z/h)}\right]\cos\Omega_1(1+\beta Z/h)\theta + \frac{(1+\beta)\gamma Z}{(1+\beta Z/h)} \quad (11)$$
$$z = Z$$

where

$$\beta = \frac{\Omega_2 - \Omega_1}{\Omega_1}$$

Using these displacement relationships, the shear stresses in the ERD, τ_{xy} and τ_{yz} become

$$\tau_{xz} = \int_0^\infty m(\theta,0)\left\{-\theta\left(\frac{\Delta\Omega}{h}\right)\left[y - \frac{(1+\beta)\gamma z}{(1+\beta z/h)}\right] + \frac{\gamma(1+\beta)}{(1+\beta z/h)^2}\sin\Omega_1(1+\beta z/h)\theta\right\}d\theta \quad (12)$$
$$= -\eta_0\left(\frac{\Delta\Omega}{h}\right)\left[y - \frac{(1+\beta)\gamma z}{(1+\beta z/h)}\right] + \gamma\Omega_1\frac{(1+\beta)}{(1+\beta z/h)}\eta'(\Omega_1 + \Delta\Omega\,z/h)$$

$$\tau_{yz} = \int_0^\infty m(\theta,0)\left\{\theta\left(\frac{\Delta\Omega}{h}\right)x + \frac{\gamma(1+\beta)}{(1+\beta z/h)^2}[1 - \cos\Omega_1(1+\beta z/h)\theta]\right\}d\theta \quad (13)$$
$$= \eta_0\left(\frac{\Delta\Omega}{h}\right)x + \frac{\gamma(1+\beta)}{(1+\beta z/h)^2}G'(\Omega_1 + \Delta\Omega\,z/h)$$

In a similar manner to the Newtonian case we balance the torques and find

$$\frac{\Delta\Omega}{\Omega_1} = 2(a/R)^2\frac{\eta'(\Omega_2)}{\eta_0} + \frac{2\,Kh}{\pi R^4\eta_0} \cong \frac{\Delta\Omega}{\Omega_2} \quad (14)$$

Comparing Equation (14) to (9) we see that for ERD lag the difference between the Newtonian and viscoelastic fluids is the ratio of the dynamic viscosity η' evaluated at the driven angular velocity over the zero shear viscosity

η_0. Since $\eta' \leq \eta_0$ lag effects should not be of any greater importance with viscoelastic fluids.

EXPERIMENT

The Rheometrics Mechanical Spectrometer (Macosko and Starita, 1971) is often operated in the ERD mode and was used to test this relationship for lag. Polarized sheets were attached to the upper and lower disks. The relative motion of the disks was studied by monitoring the change in light intensity transmitted through the polarized sheets. The lower bearing drag constant K was determined by measuring $\Delta\Omega$ at $a/R = 0$ and is the only parameter determined from experiment.

Figure 2 shows a comparison of Equation (9) to the experiments using a polydimethyl siloxane, Silicone-3 with $\eta_0 =$ 870 poise. This material is nearly Newtonian, at 15 rad/sec $\eta' = \eta_0 = 0.95$, thus the Newtonian result should model the behavior. We see that the data have the same shape as the theory: independent of a/R for small eccentricities and going with $(a/R)^2$ at larger a. The data lie parallel but somewhat below the theory at large a/R.

CONCLUSION

An evaluation of lag in the ERD geometry is important since the dynamic properties are usually determined by measuring the forces produced on the trailing disk. This study developed a relationship for lag and showed good agreement with experiment. It can safely be concluded that for polymer melts ($\eta_0 > 10^3$ poise) the lag due to bearing drag is negligible and total lag will be less than 0.5%. In fact, on our instrument with 3.6-cm radius disks, lag can probably be neglected for materials with $\eta_0 > 20$ poise.

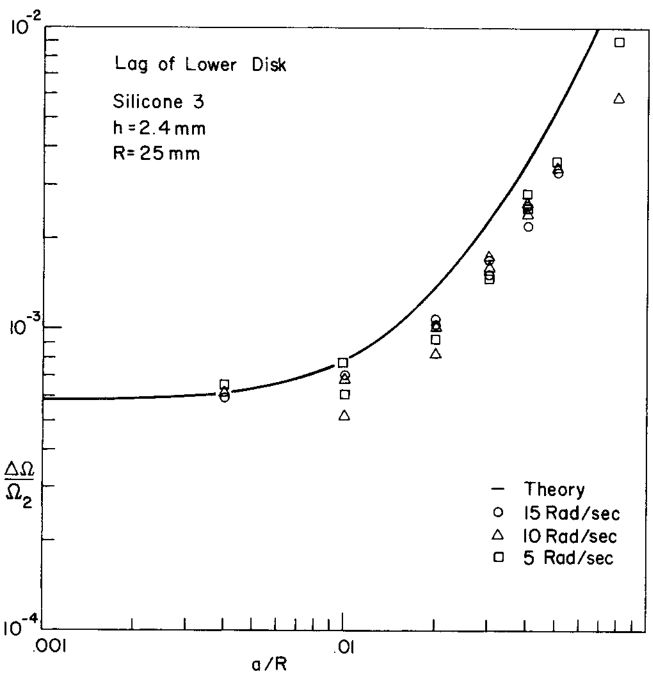

Fig. 2. Prediction of Equation (9) compared to experimental measurements of lag.

Note: Since this note was submitted the work of Payvar and Tanner (1973) has appeared which discusses mechanical equilibrium in the ERD geometry. It is gratifying that they reach similar conclusions; that the lag is small and depends on $(a/R)^2$. However, they do not appear to distinguish between F_{x1} and F_{x2}; thus the difference between Equations (8) and (9). The paper appears to make unnecessary approximations in their viscoelastic models and to neglect the effect of bearing drag. The experimental measurements are for $a > h$, a condition for which the kinematics of equations (2) and (11) are questionable. For normal experiments $a/h < 0.5$ (Gross and Maxwell, 1972; Macosko and Davis).

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NOTATION

a = eccentricity of disks
F_{x2}, F_{x1} = forces along the x axis exerted on the upper and lower disks, respectively
h = separation of disks
K = drag constant of the air bearing
M_e = external torque in the direction of the lower axis
M_i = internal torque in the direction of the lower axis
$m(\theta, 0)$ = memory function for small deformations
R = radius of disks
$\underline{\underline{S}}$ = strain tensor
t, t' = current and past time, respectively
x = fluid particle coordinate at time t
X = fluid particle coordinate at time t'

Greek Letters

β = $\dfrac{\Omega_2 - \Omega_1}{\Omega_1} = \dfrac{\Delta\Omega}{\Omega_1}$
γ = a/h
δ_{ij} = Kronecker delta
η_0 = viscosity at small shear rates
η' = dynamic viscosity
θ = $t - t'$
$\underline{\underline{\tau}}$ = stress tensor
Ω_2, Ω_1 = angular velocity of upper and lower disks, respectively
ϕ = radial angle in cylindrical coordinates

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